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The dominant of the 2-connected-Steiner-subgraph polytope for W_4-free graphs

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Abstract

This paper presents a linear-inequality description of the dominant of the polytope of the 2-connected Steiner subgraphs of a given W_4-free graph. For the special case of 2-connected spanning subgraphs, a description of the polytope is given. The latter contains the Traveling-Salesman polytope for W_4-free graphs as a face.

1. Introduction

Let $G = (V(G), E(G))$ be a 2-connected graph, and let S be a distinguished subset of vertices with $|S| \geqslant 2$. (Here, 2-connected means that G is loopless and every pair of vertices are joined by at least two internally vertex-disjoint paths.) The pair (G, S) is called a *Steiner pair*. A *Steiner subgraph* of (G, S) is a subgraph H of G such that $S \subseteq V(H)$. Let w be a real *weight* vector indexed on $E(G)$. The *weight* of a subgraph H with $E(H) \neq \emptyset$ is defined to be $\sum_{e \in E(H)} w_e$. The 2-*Connected Steiner Subgraph Problem* for (G, S, w), abbreviated 2CSSP, is that of finding a minimum-weight 2-connected Steiner subgraph. The 2CSSP *polytope* of (G, S) is the convex hull of the incidence vectors of the edge sets of 2-connected Steiner subgraphs of (G, S).

The main result of this paper is an inequality description of the dominant of the 2CSSP polytope when G is a W_4-free graph (defined in the next section). This yields

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a linear-programming formulation of the 2CSSP when G is W_4-free and $w \geqslant 0$. This also yields a description of the 2CSSP polytope for the special case of G W_4-free and $S = V(G)$. These results complement the fact that the 2CSSP on W_4-free graphs can be solved in linear time; see [7]. A description of the 2CSSP polytope for G W_4-free and S arbitrary, however, remains to be found.

The 2CSSP and several variations, the main applications of which are in the area of telecommunications-network design, have been studied structurally and algorithmically by Monma and Shallcross [17], Grötschel and Monma [12] and Grötschel et al. [13, 14]. Structural aspects of the 2CSSP when $S = V(G)$ are given in [16]. They observe that the 2CSSP is NP-hard. Additional work on the 2CSSP and variations can be found in [2, 5, 6, 15, 22]. In particular, Barahona and Mahjoub [2] describe the 2CSSP polytope for the case when $S = V(G)$ and G is a Halin graph. They also characterize the polytope with 2-connected replaced by 2-edge-connected. Coullard et al. [5] give a compact description of the 2CSSP polytope for the case when $S = V(G)$ and G is series–parallel. They [6] also describe the 2CSSP polytope when $S \subseteq V(G)$ and G is series–parallel. Mahjoub [15] characterizes the polytope when $S = V(G)$, G is series–parallel, and 2-connected is replaced by 2-edge-connected. Winter [22] describes a linear-time algorithm for the 2CSSP when G is series–parallel and $w \geqslant 0$. A survey of earlier work related to the 2CSSP is in [4].

The remainder of the paper is organized as follows. A statement of the main theorem is given in the next section. Certain graph-theoretic properties of the 2CSSP on W_4-free graphs and a number of preliminary results are given in Section 3. Section 4 presents a proof of the main theorem. Section 5 provides related results.

A basic knowledge of graph theory and polyhedral combinatorics is assumed; see, for example, [3, 20].

2. The main result

For a graph G and a vertex $v \in V(G)$, $G \backslash v$ is the graph obtained by deleting v from G. For an edge $e \in E(G)$, $G \backslash e$ is the graph obtained by deleting e from G. For a subset $K \subseteq E(G)$ and a vector y indexed on $E(G)$, $y(K)$ is defined to be $\sum_{e \in K} y_e$. Given a partition $\{V_1, \ldots, V_n\}$ of $V(G)$, $\delta_G(V_1, \ldots, V_n)$ denotes the subset of $E(G)$ containing precisely those edges whose end vertices belong to different members of the partition. For $n = 2$, $\delta_G(V_1, V_2)$ is a *(u, v)-cut* of G for any $u \in V_1$ and $v \in V_2$.

Let (G, S) be a Steiner pair. A partition $\{V_1, \ldots, V_n\}$ of $V(G)$ is *admissible* if $V_i \cap S \neq \emptyset$ and $G[V_i]$ (the subgraph induced by V_i) is connected for $i = 1, \ldots, n$.

The wheel on five vertices is denoted by W_4. A *minor* of a graph G is a graph that can be obtained from G by a sequence of edge contractions and deletions. A graph is W_4-free if it has no minor isomorphic to W_4. The class of W_4-free graphs includes that of series–parallel graphs. Barahoma and Mahjoub [1] found the stable set polytope for W_4-free graphs, and Gan and Johnson [11] studied the Chinese Postman Problem on W_4-free graphs.

For a Steiner pair (G, S), define $\boldsymbol{P}(G, S)$ to be the set of points in $\boldsymbol{R}^{E(G)}$ satisfying the following:

$$x(\delta_G(V_1, V_2)) \geqslant 2, \quad \{V_1, V_2\} \text{ admissible for } (G, S), \tag{2.1}$$

$$x(\delta_{G\backslash k}(V_1, V_2)) \geqslant 1, \quad \{V_1, V_2\} \text{ admissible for } (G\backslash k, S - \{k\}),\ k \in V(G), \tag{2.2}$$

$$x(\delta_{G\backslash k}(V_1, V_2, V_3)) \geqslant 2, \quad \{V_1, V_2, V_3\} \text{ admissible for } (G\backslash k, S - \{k\}),\ k \in V(G), \tag{2.3}$$

$$x(\delta_{G\backslash \bar{e}}(V_1, V_2)) \geqslant 1, \quad \{V_1, V_2\} \text{ admissible for } (G\backslash \bar{e}, S),\ S = \{p, q\},\ \bar{e} = pq, \tag{2.4}$$

$$x_e \geqslant 0, \quad e \in E(G). \tag{2.5}$$

For an instance (G, S, w) of the 2CSSP, define $\mathrm{LP}(G, S, w)$ as the linear-programming problem: $\min\{wx \mid x \in \boldsymbol{P}(G, S)\}$. Given a polyhedron $\boldsymbol{P}$, the *dominant* $\boldsymbol{D}$ of $\boldsymbol{P}$ is given by $\boldsymbol{D} := \{y \mid \text{there exists } x \in \boldsymbol{P} \text{ such that } y \geqslant x\}$. It is easily seen that the dominant of a polyhedron is also a polyhedron, and for $w \geqslant 0$, it is easily established that $\min\{wx \mid x \in \boldsymbol{P}\} = \min\{wx \mid x \in \boldsymbol{D}\}$. The following is the main result of the paper; its proof is given below using Theorem 2, which, in turn, is proved in Section 4.

Theorem 1. *Let (G, S) be a Steiner pair with G W_4-free. Then $\boldsymbol{P}(G, S)$ is the dominant of the 2CSSP polytope for (G, S).*

Theorem 2. *Let (G, S) be a Steiner pair with G W_4-free. Then every extreme point of $\boldsymbol{P}(G, S)$ is the incidence vector of the edge set of a 2-connected Steiner subgraph of G.*

Proof of Theorem 1. Let $\boldsymbol{D}(G, S)$ be the dominant of the 2CSSP polytope for (G, S). For $y \in \boldsymbol{D}(G, S)$, there exists an x in the 2CSSP polytope for (G, S) such that $y \geqslant x$. Evidently, every extreme point of the 2CSSP polytope belongs to $\boldsymbol{P}(G, S)$. Thus, $x \in \boldsymbol{P}(G, S)$. Moreover, since the coefficients of the inequalities of $\boldsymbol{P}(G, S)$ are all nonnegative and each inequality is $\geqslant$, it follows that $y \in \boldsymbol{P}(G, S)$. Thus, $\boldsymbol{D}(G, S) \subseteq \boldsymbol{P}(G, S)$.

Now let $y \in \boldsymbol{P}(G, S)$. Let $\{x^j\}$ for $j \in J$ be the set of extreme points of $\boldsymbol{P}(G, S)$, and let $\{r^k\}$ for $k \in K$ be the set of extreme rays of $\boldsymbol{P}(G, S)$. By Minkowski's Theorem [18], $y := x + r$ where $x := \sum_{j \in J}(\lambda_j x^j)$ and $r := \sum_{k \in K}(\mu_k r^k)$, for $\sum_{j \in J} \lambda_j = 1$ and $\lambda_j \geqslant 0$ for $j \in J$, and $\mu_k \geqslant 0$ for $k \in K$. By Theorem 2, it follows that x is in the 2CSSP polytope. Observe that $r \geqslant 0$ since $\mu_k \geqslant 0$ and $r^k \geqslant 0$ for $k \in K$. Thus, $y \in \boldsymbol{D}(G, S)$. Therefore, $\boldsymbol{P}(G, S) \subseteq \boldsymbol{D}(G, S)$. $\square$

3. Preliminary results

Let G be a 2-connected graph, and let $\{F_1, F_2\}$ be a partition of $E(G)$ such that $|F_1| \geqslant 2 \leqslant |F_2|$. If $G[F_1]$ and $G[F_2]$ have exactly two vertices in common, then each of F_1 and F_2 are *2-separators* of G. The vertices of $G[F_1]$ in common with $G[F_2]$ are the *vertices of attachment* of F_1, denoted $A(F_1)$. (Thus, if $G[F_1]$ and $G[F_2]$ each has at least three vertices, then F_1 is a 2-separator if and only if $A(F_1)$ is a 2-node cut.) If G has no 2-separator, then G is *3-connected*.

Two nonloop edges of a graph are *parallel* if they have the same set of ends. Two edges of the graph are *series* if there exists a path that contains them, every internal vertex of which as degree 2 in the graph. Assuming the convention that an edge is parallel to and in series with itself, the notions of parallel and series define equivalence relations; the associated equivalence classes are called the *parallel* and *series* classes, respectively. A parallel or series class is *nontrivial* if it has more than one edge. A graph is *simple* if it is loopless and has no nontrivial parallel class.

A *Wheatstone bridge* is the graph K_4 (the complete graph on four vertices) with any edge deleted. For a given graph G, a subgraph H of G is a *terminal* Wheatstone bridge if H is a Wheatstone bridge and the vertices of attachment of $E(H)$ are the two degree-2 vertices of H. Gan and Johnson [11] proved the following result; it also follows easily from the work of Tutte [21].

Theorem 3. *If G is a 2-connected W_4-free graph having at least five vertices, then G has a 2-separator that is either a nontrivial parallel class, a nontrivial series class, or the edge set of a terminal Wheatstone bridge.*

A 2-connected graph G is a *W-bond* if there exists distinct vertices u and v, and a partition $\{E_1, \dots, E_t\}$ $(t \geqslant 2)$ of $E(G)$ such that the vertices of attachment of each E_i are u and v, and each E_i consists of either an edge, a series class having exactly two edges, or the edge set of a terminal Wheatstone bridge.

Reductions (R1)–(R3) below are defined with respect to a Steiner pair (G, S). If G is a W_4-free graph, then in each case the resulting graph is W_4-free.

(R1) Let G be a 2-connected graph having a 2-separator F such that $(S \cap V(G[F])) - A(F) \neq \emptyset$ and $(S \cap V(G[E(G) - F])) - A(F) \neq \emptyset$. Let $A(F) = \{u, v\}$, and let G' be the graph obtained from $G[F]$ by adding a new edge e joining u and v. Set $S' := (S \cap V(G[F])) \cup \{u, v\}$.

(R2) Let G be a 2-connected graph having a 2-separator F such that $(S \cap V(G[F])) - A(F) \neq \emptyset$ and $(S \cap V(G[E(G) - F])) - A(F) = \emptyset$. Let $A(F) = \{u, v\}$, and let G' be the graph obtained from $G[F]$ by adding a new edge e joining u and v. Set $S' := S$.

(R3) Let G be a W-bond having a 2-separator F such that $E(G) - F$ is either a nontrivial series or parallel class, or the edge set of a Wheatstone bridge, and such that $S = A(F) := \{u, v\}$. Let G' be the graph obtained from $G[F]$ by adding a new edge e joining u and v. Set $S' := S$.

Define Γ to be the set of all Steiner pairs (G, S) such that G is either a bond (a connected, loopless graph on two vertices) with at most three edges, a K_3 or a K_4.

Theorem 4. *Let (G,S) be a Steiner pair with G W_4-free. If none of* (R1)–(R3) *apply to (G,S), then (G,S) belongs to Γ.*

Proof. If G is 3-connected, then it is easy to verify that (G,S) belongs to Γ (since, in this case, Theorem 3 implies G has at most four vertices). Thus, assume otherwise. Let F be a 2-separator of G. Since neither (R1) nor (R2) applies to (G,S), neither $V(G[F]) - A(F)$ nor $V(G[E(G) - F]) - A(F)$ contains a vertex of S. Thus, $|S| = 2$. Moreover, every 2-separator of G must have S as its set of vertices of attachment. Observe that the set consisting of those edges of G having at most one end in S can be partitioned into minimal 2-separators (that is, 2-separators that do not properly contain another 2-separator). To each such minimal 2-separator, add an edge joining its vertices of attachment. By minimality, the graphs so obtained are 3-connected. Moreover, they are W_4-free. By Theorem 3, they each have at most four vertices. Thus, each is either a K_3 or a K_4. It follows that G is a W-bond and, consequently, (R3) applies. □

Let (G,S) be a Steiner pair, and let (G',S') be the result of applying one of (R1)–(R3) using a 2-separator F of G. Thus, G' is the graph $G[F]$ plus a new edge e joining $A(F) := \{u,v\}$. Given $y \in \boldsymbol{P}(G,S)$, define a vector y' indexed on $E(G')$ by setting $y'_f = y_f$ for $f \in F$ and $y'_e = y(C)$, where C is a (u,v)-cut of $G[E(G) - F]$ that minimizes $y(C)$. Then, y' is *induced* by the 2-separator F.

The following lemmas use the notation established in (R1)–(R3).

Lemma 5. *Let F be a 2-separator as in* (R1), *and let $y \in \boldsymbol{P}(G,S)$. Then for any (u,v)-cut C of $G[E(G) - F]$, $y(C) \geqslant 1$.*

Proof. Observe that there exists a partition $\{V_1, V_2\}$ admissible for $(G\backslash u, S - \{u\})$, (or $G\backslash v, S - \{v\})$) such that $\delta_{G\backslash u}(V_1, V_2) \subseteq C$. By inequalities (2.2) and (2.5), $y(C) \geqslant y(\delta_{G\backslash u}(V_1, V_2)) \geqslant 1$.

Lemma 6. *Let (G,S) be a Steiner pair. Let (G',S') be obtained from (G,S) by one of* (R1)–(R3) *using a 2-separator F, and let $y \in \boldsymbol{P}(G,S)$. Moreover, in the case of* (R3), *assume that F is such that $y(C)$ is minimized over all (u,v)-cuts C of $G[E(G) - F]$ and all 2-separators of G. Then the vector y' induced by F is in $\boldsymbol{P}(G',S')$.*

Proof. Let $\{U, V\}$ be the partition of $V(G[E(G) - F])$ providing the cut C such that $u \in U$ and $v \in V$.

Consider inequalities (2.1), and let $\{V_1, V_2\}$ be admissible for (G',S'). If $\{u,v\} \subseteq V_1$, then $y'(\delta_{G'}(V_1, V_2)) = y(\delta_G(V_1 \cup V(G[E(G) - F]), V_2) \geqslant 2$ by inequalities (2.1) (since $\{V_1 \cup V(G[E(G) - F]), V_2\}$ is admissible for (G,S) and $\delta_{G'}(V_1, V_2) = \delta_G(V_1 \cup V(G[E(G) - F]), V_2)$).

Now suppose $u \in V_1$ and $v \in V_2$. If (R1) has been applied, then $y'(\delta_{G'}(V_1, V_2)) \geqslant 2$ by Lemma 5 and the definition of y'_e. Otherwise, $y'(\delta_{G'}(V_1, V_2)) = y(\delta_G(V_1 \cup U, V_2 \cup V)) \geqslant 2$ by the definition of y'_e and inequalities (2.1).

Consider inequalities (2.2), and let $\{V_1, V_2\}$ be admissible for $(G'\backslash k, S' - \{k\})$. If $\{u,v\} \subseteq V_1 \cup \{k\}$, then $y'(\delta_{G'\backslash k}(V_1, V_2)) = y(\delta_{G\backslash k}(V_1 \cup (V(G[E(G)-F]) - \{k\}), V_2)) \geqslant 1$ by inequalities (2.2).

Now suppose $u \in V_1$ and $v \in V_2$. If (R1) has been applied, then $y'(\delta_{G'\backslash k}(V_1, V_2) \geqslant y'_e \geqslant 1$ by Lemma 5 and the definition of y'_e. Otherwise, $y'(\delta_{G'\backslash k}(V_1, V_2)) = y(\delta_{G\backslash k}(V_1 \cup U, V_2 \cup V)) \geqslant 1$ by the definition of y'_e and inequalities (2.2).

Consider inequalities (2.3), and let $\{V_1, V_2, V_3\}$ be admissible for $(G'\backslash k, S' - \{k\})$. If $\{u,v\} \subseteq V_1 \cup \{k\}$, then $y'(\delta_{G'\backslash k}(V_1, V_2, V_3)) = y(\delta_{G\backslash k}(V_1 \cup (V(G[E(G)-F]) - \{k\}), V_2, V_3)) \geqslant 2$ by inequalities (2.3).

Now suppose $u \in V_1$ and $v \in V_2$. If (R1) has been applied, $y'_e \geqslant 1$ by Lemma 5 and the definition of y'_e. Moreover, $\delta_{G\backslash k}(V_1 \cup V_2 \cup V(G[E(G)-F]), V_3) \subseteq \delta_{G'\backslash k}(V_1, V_2, V_3) - \{e\}$, and so $y'(\delta_{G'\backslash k}(V_1, V_2, V_3) - \{e\}) \geqslant y(\delta_{G\backslash k}(V_1 \cup V_2 \cup V(G[E(G)-F]), V_3)) \geqslant 1$ by inequalities (2.2). Thus, $y'(\delta_{G'\backslash k}(V_1, V_2, V_3)) \geqslant 2$. Otherwise, in the case that either (R2) or (R3) has been applied, $y'(\delta_{G'\backslash k}(V_1, V_2, V_3)) = y(\delta_{G\backslash k}(V_1 \cup U, V_2 \cup V, V_3)) \geqslant 2$ by the definition of y'_e and inequalities (2.3).

Consider inequalities (2.4), and let $\{V_1, V_2\}$ be admissible for $(G'\backslash \bar{e}, S')$, where $S' = \{p, q\}$ and $\bar{e} = pq$. First, suppose $\bar{e} \neq e$. If $\{u,v\} \subseteq V_1$, then $y'(\delta_{G'\backslash \bar{e}}(V_1, V_2)) = y(\delta_{G\backslash \bar{e}}(V_1 \cup V(G[E(G)-F]), V_2)) \geqslant 1$ by inequalities (2.4). If $u \in V_1$ and $v \in V_2$, then $y'(\delta_{G'\backslash \bar{e}}(V_1, V_2)) = y(\delta_{G\backslash \bar{e}}(V_1 \cup U, V_2 \cup V)) \geqslant 1$ by the definition of y'_e and inequalities (2.4).

Now, suppose that $e = \bar{e}$. If $y'(\delta_{G'\backslash \bar{e}}(V_1, V_2)) < 1$, then since F was chosen so that $y(C)$ is minimized over all (u,v)-cuts of $G[E(G) - F]$, $y(\delta_G(V_1 \cup U, V_2 \cup V)) < 2$, contradicting inequalities (2.1).

Inequalities (2.5) are evidently satisfied by y'. □

4. Proof of the main result

This section gives a proof of Theorem 2. The proof uses the following result, the proof of which is somewhat tedious and is postponed to Section 6.

Lemma 7. *Let (G, S) be a Steiner pair with $(G, S) \in \Gamma$. Then every extreme point of $P(G, S)$ is the incidence vector of the edge set of a 2-connected Steiner subgraph of (G, S).*

Proof of Theorem 2. Evidently, $\mathrm{LP}(G, S, w)$ is feasible and not unbounded for any $w \geqslant 0$. Moreover, for any w that is not nonnegative, $\mathrm{LP}(G, S, w)$ is unbounded. Thus, it suffices to show that for any $w \geqslant 0$, $\mathrm{LP}(G, S, w)$ has an optimal solution that is the incidence vector of the edge set of a 2-connected Steiner subgraph of (G, S). To this end, let y be an optimal solution to LP(G, S, w). The proof uses induction of $|E(G)|$. If $(G, S) \in \Gamma$, then the result is given by Lemma 7. If this is not the case, then, by Theorem 4, one of the reductions (R1)–(R3) applies.

Suppose (G_1, S_1) and (G_2, S_2) are obtained from (G, S) by (R1) using 2-separators F_1 and $F_2 := E(G) - F_1$, respectively. Define w^1 by $w_e^1 := 0$ and $w_f^1 := w_f$ for $f \in F_1$; define w^2 analogously. Let x^1 and x^2 be an extreme-point optimal solutions to LP(G_1, S_2, w^1) and LP(G_2, S_2, w^2), respectively. Define $\bar{x}$ indexed on $E(G)$ by $\bar{x}_f := x_f^1$ for $f \in F_1$ and $\bar{x}_f := x_f^2$ for $f \in F_2$. Then, $w\bar{x} = w^1x^1 + w^2x^2$ by definition of w^1 and w^2. By induction, the graph induced by the support of $\bar{x}$ is a 2-connected Steiner subgraph of (G, S). Thus, $\bar{x}$ is feasible to LP(G, S, w).

If $\bar{x}$ is not optimal to LP(G, S, w), then $wy < w\bar{x}$. By Lemma 6, there are feasible solutions y^1 and y^2 to LP(G_1, S_1, w^1) and LP(G_2, S_2, w^2), respectively, such that $wy = w^1y^1 + w^2y^2$, which implies $w^1y^1 + w^2y^2 < w^1x^1 + w^2x^2$, contradicting the optimality of either x^1 or x^2.

Suppose now that (G', S') is obtained from (G, S) by (R2) or (R3) using a 2-separator F. Moreover, in the case (R3), assume that F is such that $y(C)$ is minimized over all (u, v)-cuts C of $G[E(G) - F]$ and all 2-separators of G. Define w' by $w'_e := w(P^*)$, where P^* is a minimum-weight (u, v)-path in $G[E(G) - F]$, and $w'_f := w_f$ for $f \in F$. Let x' be an extreme-point optimal solution to LP(G', S', w'). Define $\bar{x}$ by setting $\bar{x}_f := x'_f$ for all $f \in F$, $\bar{x}_f := x'_e$ for $f \in P^*$, and $\bar{x}_f := 0$ for $f \in E(G) - (F \cup P^*)$. Thus, $w\bar{x} = w'x'$ by definition of w'. By induction, the graph induced by the support of $\bar{x}$ is a 2-connected Steiner subgraph of (G, S). Thus, $\bar{x}$ is feasible to LP(G, S, w).

If $\bar{x}$ is not optimal to LP(G, S, w), then $wy < w\bar{x}$. By Lemma 6, there is a feasible solution to y' to LP(G', S', w') that is induced by F. By the length–width inequality [8, 10] and the definition of w', $w'_e y'_e \leqslant \sum_{f \in E(G) - F} w_f y_f$, and so, $w'y' \leqslant wy$, which implies $w'y' < w'x'$, contradicting the optimality of x'. □

5. Related results

A subgraph H of a graph G is *spanning* if $V(H) = V(G)$. The *2-connected-spanning-subgraph polytope* of G is the 2-connected-Steiner-subgraph polytope for the Steiner pair $(G, V(G))$.

The dominant of the 2-connected-spanning-subgraph polytope for a W_4-free graph G is evidently given by inequalities (2.1)–(2.5) with $S = V(G)$. It turns out that a description of the polytope itself is obtained by simply adding the inequalities $x_e \leqslant 1$ for $e \in E(G)$. This follows from a result of Rais [19] that characterizes precisely when a polytope and its dominant are related in this way. This simple relationship does not hold for the more general 2CSSP polytope; see [19] for details.

Observe that a 2-connected subgraph is a cycle if and only if the number of edges in the subgraph is equal to the number of vertices. It follows that the traveling-salesman polytope is obtained from the 2-connected-spanning-subgraph polytope by adding the equation $x(E(G)) = |V(G)|$.

Finally, series–parallel graphs are W_4-free since they are characterized as having no K_4; see [9]. If G is series–parallel, then the description of the dominant can be simplified. In particular, inequalities (2.3) are not necessary.

6. Proof of Lemma 7

This section presents a proof of Lemma 7. In particular, if $G = K_4$, then Lemma 7 follows from Lemma 9 below, and if G is a bond or $G = K_3$, then Lemma 7 follows from Lemma 8 below and inequalities (2.1).

Lemma 8. *Let (G, S) be a Steiner pair, and let $\bar{x}$ be an extreme point of $P(G, S)$. Then $\bar{x}_e \leqslant 1$ for all $e \in E(G)$.*

Proof. Define a vector y by setting $y_e := 1$ for each $e \in E(G)$ such that $\bar{x}_e > 1$ and $y_e := \bar{x}_e$ otherwise. Let $\bar{x}$ be the unique optimal solution to $\mathrm{LP}(G, S, w)$. Then, $w \geqslant 0$ for otherwise $\mathrm{LP}(G, S, w)$ is unbounded. Thus, $wy \leqslant w\bar{x}$. The result now follows by showing that y is feasible to $\mathrm{LP}(G, S, w)$. Note y satisfies inequalities (2.2), (2.4), and (2.5).

Consider inequalities (2.1). Let $\{V_1, V_2\}$ be admissible to (G, S), and suppose $y(\delta_G(V_1, V_2)) < 2$. Then $\delta_G(V_1, V_2)$ contains exactly one edge, say $e = uv$, such that $y_e = 1 < \bar{x}_e$. Assume $u \in V_1$. If $S = \{u, v\}$, then $y(\delta_G(V_1, V_2)) = y(\delta_{G \backslash e}(V_1, V_2)) + y_e \geqslant 2$ by inequalities (2.4) and the definition of y_e. If $S \neq \{u, v\}$ then, without loss of generality, assume $(V_1 \cap S) - \{u\} \neq \emptyset$. Note that $\{V_1 - \{u\}, V_2\}$ is admissible for $(G \backslash u, S - \{u\})$ and $\delta_{G \backslash u}(V_1 - \{u\}, V_2) \subseteq \delta_G(V_1, V_2) - \{e\}$. Thus, $y(\delta_G(V_1, V_2)) \geqslant y(\delta_{G \backslash u}(V_1 - \{u\}, V_2)) + y_e \geqslant 2$ by inequalities (2.2) and (2.5) and the definition of y_e.

Consider inequalities (2.3). Let $\{V_1, V_2, V_3\}$ be admissible for $(G \backslash k, S - \{k\})$, and suppose $y(\delta_{G \backslash k}(V_1, V_2, V_3)) < 2$. Then $\delta_{G \backslash k}(V_1, V_2, V_3)$ contains exactly one edge, say $e = uv$, such that $y_e = 1 < \bar{x}_e$. Assume $u \in V_1$ and $v \in V_2$. Note that $\{V_1 \cup V_2, V_3\}$ is admissible for $(G \backslash k, S - \{k\})$ and $\delta_{G \backslash k}(V_1 \cup V_2, V_3) \subseteq \delta_{G \backslash k}(V_1, V_2, V_3) - \{e\}$. Thus, $y(\delta_{G \backslash k}(V_1, V_2, V_3)) \geqslant y(\delta_{G \backslash k}(V_1 \cup V_2, V_3)) + y_e \geqslant 2$ by inequalities (2.2) and (2.5) and the definition of y_e. $\square$

Lemma 9. *Let (G, S) be a Steiner pair with $G = K_4$, and let $\bar{x}$ be an extreme point of $P(G, S)$. Then $\bar{x}$ is the incidence vector of the edge set of a 2-connected Steiner subgraph of (G, S).*

Proof. Let $V(G) := \{p, q, r, s\}$ with $S := \{p, q\}$ if $|S| = 2$ and $S := \{p, q, r\}$ if $|S| = 3$. Let the edges of G be $e := pq$, $f := pr$, $g := ps$, $h := qr$, $i := qs$, and $j := rs$. Let $\bar{x}$ be the unique optimal solution to $\mathrm{LP}(G, S, w)$. Then, $w \geqslant 0$. By Lemma 8, $\bar{x} \leqslant 1$.

Since $\bar{x}$ is an extreme point and $|E(G)| = 6$, there exist at least six inequalities among (2.1)–(2.5) that are tight with respect to $\bar{x}$; that is, are satisfied with equality.

Suppose $\bar{x}_e = 0$. Then $\bar{x}_f = \bar{x}_g = \bar{x}_h = \bar{x}_i = 1$ by inequalities (2.1). Moreover, $\bar{x}_j = 0$ for otherwise reducing $\bar{x}_j$ by sufficiently small positive amount results in a feasible solution having objective less than or equal to $\bar{x}$, a contradiction. Thus, $\bar{x}$ is the incidence vector of the edge set of a 2-connected Steiner subgraph of (G, S). By symmetry, assume $\bar{x}_t > 0$ for all edges having both ends in S.

For an ε to be specified, define x^1 by $x^1_e := \bar{x}_e + \varepsilon$, $x^1_f := \bar{x}_f - \varepsilon$, $x^1_j := \bar{x}_j + \varepsilon$, $x^1_i := \bar{x}_i - \varepsilon$, and $x^1_t := \bar{x}_t$ for $t \in \{g, h\}$. Define x^2 so that $\bar{x} = \frac{1}{2}(x^1 + x^2)$. Also, define x^3 by $x^3_e := \bar{x}_e + \varepsilon$, $x^3_g := \bar{x}_g - \varepsilon$, $x^3_i := \bar{x}_i - \varepsilon$, and $x^3_t := \bar{x}_t$ for $t \in \{f, h, j\}$. Finally, define x^4 so that $\bar{x} = \frac{1}{2}(x^3 + x^4)$.

First, suppose that $|S| = 4$. Observe that the (2.1) inequalities in which $|V_1| = |V_2| = 2$ are implied by inequalities (2.2). If $\bar{x}_g, \bar{x}_i, \bar{x}_j < 1$, then none of the (2.2) inequalities are tight with respect to $\bar{x}$. As a consequence, x^1 and x^2, as defined above, satisfy (2.1)–(2.5) for sufficiently small ε, a contradiction to $\bar{x}$ being an extreme point. Thus, by symmetry, it can be assumed that either $\bar{x}_g = \bar{x}_h = 1$ or $\bar{x}_g = \bar{x}_i = \bar{x}_j = 1$. In the former case, again it can be seen that x^1 and x^2 satisfy (2.1)–(2.5) for sufficiently small ε. In the latter case, none of the (2.2) or (2.3) inequalities in which x_g, x_i, or x_j have a nonzero coefficient are tight. Thus, the (2.1) inequalities $x_e + x_f + x_g \geqslant 2$, $x_e + x_h + x_i \geqslant 2$, and $x_f + x_h + x_j \geqslant 2$ are tight (since each variable must have a nonzero coefficient in some tight inequality). This in turn implies that the (2.2) inequalities $x_e + x_f \geqslant 1$, $x_e + x_h \geqslant 1$, and $x_f + x_h \geqslant 1$ are tight. These three equations, however, contradict the (2.3) inequality $x_e + x_f + x_h \geqslant 2$.

Now suppose that $|S| = 2$ or 3. If $|S| = 3$, then the (2.1) inequalities in which $|V_1| = |V_2| = 2$ are implied by the (2.2) inequalities.

If $\bar{x}_e = \bar{x}_f = \bar{x}_h = 1$, then $\bar{x}_g = \bar{x}_i = \bar{x}_j = 0$ for otherwise there are no tight inequalities. Thus, $\bar{x}$ is the incidence vector of a 2-connected Steiner subgraph.

If $\bar{x}_e < 1$ and $\bar{x}_f = \bar{x}_h = 1$, then $\bar{x}_g, \bar{x}_i > 0$ by the (2.2) inequalities. It follows that x^3 and x^4 as defined above, satisfy (2.1)–(2.5) for sufficiently small ε, a contradiction.

If $\bar{x}_e, \bar{x}_f < 1$ and $\bar{x}_h = 1$, then $\bar{x}_f, \bar{x}_g, \bar{x}_i > 0$ by the (2.2) inequalities. Here the cases of $|S| = 2$ and $|S| = 3$ are separated.

If $|S| = 2$, then no (2.4) inequalities are tight. Also, the (2.2) inequalities $x_e + x_g \geqslant 1$ and $x_e + x_h \geqslant 1$ are not tight. This leaves seven potentially tight inequalities. If $\bar{x}_g < 1$, then the (2.2) inequality $x_e + x_f \geqslant 1$ is not tight. On the other hand, if $\bar{x}_g = 1$, then the (2.1) inequality $x_e + x_g + x_h + x_j \geqslant 2$ is not tight. Thus, in either case, the (2.1) inequality $x_e + x_f + x_i + x_j \geqslant 2$, the (2.2) inequality $x_e + x_i \geqslant 1$, and the non-negativity inequality $x_j \geqslant 0$ are all tight. These three equations imply that $\bar{x}_f = 1$, a contradiction.

If $|S| = 3$, then $\bar{x}_j > 0$ by the (2.2) inequalities. None of the (2.2) inequalities, with the possible exception of $x_e + x_f \geqslant 1$, $x_e + x_i \geqslant 1$, and $x_f + x_j \geqslant 1$, are tight. This leaves seven potentially tight inequalities (not counting the redundant (2.1) inequalities). If $\bar{x}_g < 1$, then the (2.2) inequality $x_e + x_f \geqslant 1$ is not tight, which implies that the (2.1) inequality $x_e + x_f + x_g \geqslant 2$ and the (2.3) inequality $x_e + x_f + x_h \geqslant 2$ are tight.

This implies $\bar{x}_g = 1$, a contradiction. Thus, $\bar{x}_g = 1$. It follows that x^1 and x^2 satisfy (2.1)–(2.5) for sufficiently small ε, a contradiction.

If $\bar{x}_e, \bar{x}_f, \bar{x}_h < 1$, then $\bar{x}_f, \bar{x}_h, \bar{x}_g, \bar{x}_i > 0$ by the (2.2) inequalities. The $|S| = 2$ and $|S| = 3$ cases are again separated.

If $|S| = 2$, then no (2.4) inequalities are tight. Also, the (2.2) inequalities $x_e + x_g \geqslant 1$ and $x_e + x_i \geqslant 1$ are not tight. If $\bar{x}_g < 1$, then $x_e + x_f \geqslant 1$ is not tight, implying that $x_e + x_g + x_h + x_j \geqslant 2$, $x_e + x_h \geqslant 1$, and $x_j \geqslant 0$ are all tight. These three equations contradict that the fact that $\bar{x}_g < 1$. Thus, $\bar{x}_g = 1$, and by symmetry this case now reduces to a previous one.

If $|S| = 3$, then $\bar{x}_j > 0$ by the (2.2) inequalities. Moreover, no (2.2) inequalities are tight. It follows that x^1 and x^2 satisfy (2.1)–(2.5) for sufficiently small ε, a contradiction. □

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